The Harmful Effects of “Carrying” and “Borrowing” in Grades 1-4*

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In the school where we conducted our research, some teachers taught “carrying” and “borrowing,” but others did not. As can be seen in the following list, the teachers in the upper grades tended to teach these conventional United States algorithms:

Grade l: None of the four teachers taught any algorithms.

Grade 2: One of the three teachers taught algorithms.

Grade 3: Two of the three teachers taught algorithms.

Grade 4: All four teachers taught algorithms.

Because the principal mixed all the children at each grade level before each school year and divided them as randomly as possible, all the classes were heterogeneous and comparable.

At the end of a school year, we asked 185 children in grades 2-4 to solve 7 + 52 + 186 (or 6 + 53 + 185) in individual interviews. Note that this problem had a one-digit number, a two-digit number, and a three-digit number. The children were allowed to look at the problem as long as they wanted to, but they were not permitted to write anything. The answers given by three classes of second graders are presented in Table 1. The first column titled “Algorithms” shows the answers given by the class who had been taught the algorithms. In the two other classes, the teachers did not teach the algorithms, but only the last column is titled “No algorithms.” The

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*A more complete account of this study entitled “The Harmful Effects of Algorithms in Grades 1-4” can be found in Chapter 17 of the l998 NCTM Yearbook (Kamii & Dominick, 1998). The National Council of Teachers of Mathematics holds the copyright to this chapter and permitted the reprinting only of the tables for the present article.

reason is that when parents taught the algorithms at home, only one of the teachers called them to ask them to come to school to observe their children. The column in the middle of Table 1 is titled “Some algorithms taught at home” because the teacher did not teach these rules but did not try to convince parents that this teaching was harmful.

Table 1

Answers to 7 + 52 + 186 Given by Three Classes of Second Graders*

Algorithms *n* = 17	Some algorithms taught at home *n* = 19	No algorithms *n* = 20
9308		
1000		
989		
986		
938	989	
906	938	
838	810	617
295	356	255
		246
245 (12%)	245 (26%)	245 (45%)
		243
		236
		235
200	213	138
198	213	---**
30	199	---**
29	133	---**
29	125	---**
---**	114	
---**	---**	
	---**	
	---**	

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**--- indicates that the child did not work the problem.

It can be seen in Table 1 that the correct answer of 245 was given by 12% of the "Algorithms" class, 26% of the "Some algorithms" class, and 45% of the "No algorithms" class. More significant than the proportions getting the correct answer are the wrong answers the children gave. All the wrong answers given in each class are listed in Table 1. It can be observed that the incorrect answers given by the "No algorithms" class were much more reasonable than those given by the "Algorithms" class. In the "No algorithms" class, the wrong answers ranged from 138 to 617, but the range in the "Algorithms" class was from 29 to 9308. The "Some algorithms" class came out in between, with respect to the correct answer as well as to the wrong answers.

Table 2 gives the findings from three third-grade classes. There were about 20 students in the "No algorithms" class, but only 10 of them are included in Table 2 because there were only 10 students who had never been taught to use algorithms. It can be observed in Table 2 that the "No algorithms" class again had the highest percentage of the correct answer (50% vs. 32% and 20%), and the wrong answers the "No algorithms" class gave were much more reasonable than in the "Algorithms" classes. The range of wrong answers in the "No algorithms" class was from 221 to 284, but it was from 29 to 838 in the "Algorithms" classes.

Table 3 presents the data about fourth grade. All four of the fourth-grade teachers taught algorithms, and two points can be made based on Table 3. First, the percentage of fourth graders getting the correct answer (about 20% on average) was lower than that of the second graders who were not taught any algorithms (45%). Second, a new phenomenon emerged in fourth grade: answers like "4, 4, 4" and "1, 3, 2." These answers show that the students were thinking about the total, such as 132, as three separate one-digit numbers ("one," "three," and "two").

Table 2

Answers to 6 + 53 + 185 Given by Three Classes of Third Graders*

Algorithms *n* = 19	Algorithms *n* = 20	No algorithms *n* = 10
	800 + 38	
838	800	
768	444	
533	344	284
246		245
244 (32%)	244 (20%)	244 (50%)
235	243	243
234	239	238
213	238	221
194	234	
194	204	
74	202	
29	190	
---**	187	
---**	144	
	139	
	---**	
	---**	

*Reprinted, with permission, from the 1998 NCTM Yearbook, *The Teaching and Learning of Algorithms in School Mathematics,*

**--- indicates that the child did not work the problem.

Why Are Algorithms Harmful?

Algorithms are harmful to most young children for two reasons: (1) They encourage children to give up their own thinking, and (2) they "unteach" what children know about place value, thereby preventing them from developing number sense.

As early as 1985, Madell (1985) made the following statement based on research he conducted in a private school in New York City: When children are allowed to do their own thinking, "they universally proceed from left to right (p. 21)." To do 36 + 46, for example,

Table 3

Answers to 6 + 53 + 185 Given by Four Classes of Fourth Graders*

Algorithms *n* = 20	Algorithms *n* = 21	Algorithms *n = 21*	Algorithms *n = 18*
	1215		
	848		
	844		
	783		
1300	783		10,099
814	783		838
744	718	791	835
715	713	738	745
713 + 8	445	721	274
	245		
244 (30%)	244 (24%)	244 (19%)	244 (17%)
243	234		234
	224		234
			234
194	194	144	225
177	127	138	8, 3, 8
144	---*	134	4, 3, 2
143	---*	8, 3, 7	4, 3, 2
134		8, 1, 7	---**
4, 4, 4		---**	---**
1, 3, 2		---**	---**
---**		---**	
		---**	
		---**	
		---**	
		---**	
		---**	
		---**	

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**--- indicates that the child did not work the problem.

children naturally add 30 + 40 = 70 first, then 6 + 6 = 12, and finally 70 + 12 = 82. If they are taught to use the conventional U. S. algorithm, they add from right to left, which goes counter to their natural thinking. Children thus give up their own thinking, and because they have given up their own thinking, these children are not bothered by getting answers like 29 for 7 + 52 + 186.

The conventional algorithm makes children treat every column as a column of ones. For example, if we listen to them while they are using the algorithm to solve

$$\begin{array}{r} 136 \\ \underline{+246} \end{array},$$

we can hear them saying "Six and six is twelve. Put down the two and carry the **one. One** and **three** and **four** is **eight**. . . **One** and **two** is **three**. . . ." Treating every column as a column of ones is convenient for adults, who already know solidly that the "3" in "136" means 30. For young children who are still trying to learn place value, however, the conventional algorithms serve to "unteach" place value. The second graders in Table 1 who got 29 for 7 + 52 + 186 got this answer by adding 7 + 5 + 2 + 1 + 8 + 6. Those who got answers in the 900s got them by getting confused about place value. Typically, they started by saying "7 + 2 + 6 = 15. Put down the 5 and carry the one. One and 5 and 8 is 14. Oh! I forgot what I put down here. I'll start over . . . 7 + 1 is 8, and I have to 'carry' the 1 (from 5 + 8) . . . so the 8 becomes a 9. . . ." Answers like "4, 4, 4," too, reflect children's thinking about every column as a column of ones.

By contrast, most of the children in the "No algorithms" classes typically began by saying, "A hundred eighty and fifty is two hundred thirty." This is why even if they made errors, the answers of the "No algorithms" classes were mostly between 230 and 260. The second grader in the "No algorithms" class who got the answer of 617 had been regularly coached at home, even though her parents had said that they would stop this teaching.

Not All Children Are Harmed by Algorithms.

In Tables 1, 2 and 3, there are small percentages of children who were taught algorithms but still got correct answers (12% in second grade, about 25% on average in third grade, and about 20% on average in fourth grade). There is usually a minority of children in every class who are cognitively more advanced than the others and can easily operate on numbers from left to right, as well as from right to left. Even in first grade, there are a few children who figure out place value on their own, before instruction is given. These are the children who probably go on to advanced sections of mathematics in high school and often get graduate degrees in mathematics. Today's physicists, engineers, and professors of mathematics were probably these kinds of children to whom school mathematics made sense. What we have observed, however, is that the type of instruction children are usually given makes sense only to a small minority.

The Constructive Process in History

Many historical algorithms show a parallel between a child's construction of arithmetic and our ancestors' construction. For example, some Hindus added 278 and 356 on a "dust" board before pen and paper were invented, and the digits were erased as they were added. Groza (1968) presented the following example:

278--------578--------628--------634

356   56   6

In this procedure, the 300 was first added to 200. The top number thus became 578, and the 3 of 356 was erased. Next, 50 was added to 70. The top number thus became 628, and the 5 of 56 was erased. The 6 was added to 8 last; the total became 634; and the 6 was erased. Because this left-to-right process required much writing and erasing, mathematicians later invented "carrying" and "borrowing" that became the conventional methods taught in most U. S. schools today.

It took centuries for mathematicians to invent, or construct, “carrying” and “borrowing.” When we teach these algorithms to children without letting them go through a left-to-right process, we are requiring them to skip a step in their development. Babies need to crawl before they walk (although a few walk without crawling). Most say “Ball gone” before they say “The ball is gone.” Teaching them to “carry” and to “borrow” makes children skip a stage of development that took centuries for adult mathematicians to invent. Since 1972, Ashlock (1972, 1976, 1982, 1986, 1990, 1994, 1998, 2002, 2006. 2010) has been publishing an astonishing variety of data showing that children do not understand “carrying,” “borrowing,” and the many other computational rules they have been taught in school.

The point of teaching children to add and subtract should not be to teach them to get correct answers by using the U. S. algorithms. It should be to encourage them to think clearly and logically to solve problems. To find the duration between 8:49 a.m. and 10:35 a.m., for example, some children “borrow” ten as can be seen in Figure 1, after being taught the algorithm shown in Figure 2. (These children first “borrow” ten from the 3 and do 15 – 9 = 6. They then “borrow” ten from 10 and do 12 – 4 = 8. Because 9 – 8 = 1, they get the final answer of 1 hour 86 minutes.)

Figure 1. The Algorithm of “Borrowing” Used by Some Children

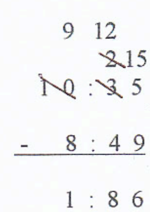
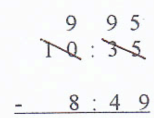


Figure 2. The Algorithm of "Borrowing" Presented in Some Textbooks



Most adults would use addition in the following way to answer this question: From 8:49 to 9:49 is 1 hour, and from 9:49 to 10:35 is 46 minutes. So the answer is 1 hour and 46 minutes. Children in grades 2-4 also use addition but usually reason as follows:

From 8:49 to 9:00 is 11 minutes.

From 9:00 to 10:00 is 1 hour.

From 10:00 to 10:35 is 35 minutes.

So the answer is 1hour and 46 minutes.

Children who do their own thinking are not confused by "borrowing" and naturally rearrange quantities in a variety of ways depending on the problem. This freedom to do one's own thinking also leads to the use of the distributive property which will be helpful for solving more complex problems later on. If children are asked to make sense of problems, they are not limited to learning isolated skills, and they also learn about mathematical relationships and how those relationships can be used to solve problems.

The Constructive Process in Education

Most professions seem to go through a constructive process. Agriculture developed enormously during the 1900s through scientific research. Likewise, engineering, medicine, and architecture keep making progress from year to year through scientific research. Education, by contrast, seems to keep going back to methods that have not worked in spite of research to the contrary.

"The Harmful Effects of Algorithms in Grades 1-4" was published in 1998, four years after Kamii (1994) had published even more data. But 15 years later, most curricula still include the teaching of "carrying" and "borrowing." When educators use research to inform practice and teach mathematics as a sense-making discipline, we will have a much better chance of helping all children be successful in mathematics.

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