Activity 2: Introduction to Probability & Counting

1. In the last activity, we flipped coins under the assumption that we had a 50% chance of getting either heads or tails. If that assumption is true, why don't we expect to obtain exactly 50 heads if we toss a coin 100 times? How can we even be sure we have a 50% chance of flipping heads on any single coin toss?

> A phenomenon is **random** if individual outcomes are uncertain but there is nonetheless a regular distribution of outcomes in a large number of repetitions.
>
> The **probability** of any outcome refers to our prediction of the likelihood of that outcome occurring in the future.

2. What is the probability that I am reading this question on the second day of class? ____

   What is the probability that I will read this question on the second day of class next year? __________________________________

   What is the probability that you showed up to class today? ____

> One of our primary goals for this first unit is to learn how to develop probability models.
>
> A **probability model** displays (graphically, via formulas, or through a list):
> - The **sample space** of an experiment (all possible outcomes of that experiment)
> - The **probability** of observing each of those outcomes

3. Write out the probability model for tossing a fair, six-sided die.

   Write out the probability model for your grade in this course. How did you estimate the probabilities for each model?

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Probabilities are extremely useful, but they can be interpreted in many ways. Physical probabilities, such as those arising from coin tosses, card games, or radioactive atoms, can be interpreted and estimated a couple different ways:

Relative frequency approach: Probability is how frequently something happens over a large number of repetitions
Classical approach: Probability is related directly to the number of possible outcomes in an experiment.

Evidential probabilities, such as the our beliefs in the fairness of a coin or the probability that it will rain tomorrow, can be estimated through a subjective approach:

Subjective approach: Probability represents our degree of belief in an outcome based on previous evidence.

Relative Frequency Approach: $P(\text{win}) \approx \lim_{\text{trials} \to \infty} \frac{\text{wins}}{\text{trials}}$

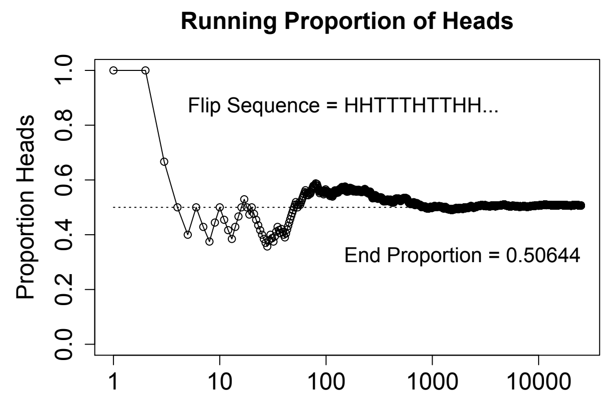
We estimate probability as the proportion of times an outcome occurs in a long series of repetitions.

4. Suppose we toss a coin 10 times and observe 7 heads. According to the relative frequency approach, our best estimate for the probability of tossing heads would be 7/10 = 0.70. Why is this estimate inaccurate?

5. Karl Pearson, a famous statistician, once tossed a coin 24,000 times and observed 12,012 heads. Using his results, we'd estimate the probability of tossing heads to be 0.5005.

   We can use a computer to simulate Pearson's experiment. The graph shows the running proportion of heads when a simulated coin is flipped 25,000 times. While the outcome of each coin flip is random, the long-run relative frequency settled right near 0.50.

   Under the relative frequency approach, a greater number of repetitions yields a probability estimate that is more stable.



   Using the relative frequency approach, how would you estimate the following probabilities?

   a) What's the probability that the current vice president will become president in the future?

   b) What's the probability that Kobe Bryant makes his next free throw?

   c) If we roll two dice and calculate their sum, what's the probability that the sum is 7? To estimate this probability, I had a computer simulate 5,000 rolls of two dice. From these 5,000 rolls, 897 summed to 7. From this information, what's our best estimate for the probability of rolling two dice and getting a sum of 7?

   Can you think of another way to estimate the probability of rolling a 7?

Classical / Theoretical Approach:

If an experiment has S outcomes and **each outcome is equally likely to occur**, then the probability of event A is:

$$P(\mathrm{A}) = \frac{\text{number of A outcomes}}{\text{number of S outcomes}}$$

6. To estimate probabilities using this approach, we must be able to count the total number of possible outcomes. Suppose we toss a coin 3 times. List the sample space (all possible outcomes). Is each outcome in the sample space equally likely to occur? What's the probability that we observe at least 2 heads?

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7. Again, suppose we toss a coin 3 times, but we're only interested in the **number of heads** we observe. List the sample space. Is each outcome in the sample space equally likely to occur? What's the probability that we observe at least 2 HEADS?

8. The following table shows the 36 possible outcomes from rolling 2 dice. Is each outcome equally likely? What's the probability of rolling a sum of 12? What's the probability of rolling a sum of 7? How does this probability compare to what we calculated under the relative frequency approach?

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

9. The classical approach is useful when estimating probabilities (**only if all outcomes are equally likely**), but counting outcomes can quickly become tedious. For example:

Experiment	Number of outcomes
Rolling a 6-sided die 1 time	6
Rolling a 6-sided die 2 times	36
Rolling a 6-sided die 3 times	216
Rolling a 6-sided die 10 times	60,466,176
Tossing a coin 10 times	1,024
Tossing a coin 20 times	1,048,576
Randomly dividing 20 students into 2 groups of 10 students	184,756 (if group identification matters)
Randomly dividing 20 students into 4 groups of 5 students	11,732,745,024 (if group identification matters)
Choosing 6 lottery numbers between 1-36 with no repeats	1,402,410,240
Choosing 6 lottery numbers between 1-36 with repeats	2,176,782,336

Remember, to estimate probabilities using the classical approach, we need to be able to count all possible outcomes of an experiment (we do NOT, thankfully, need to list them all out).

To help count all possible outcomes in an experiment, we'll learn several counting rules. Because these rules are so useful, we will try to gain a deep understanding of them. When it comes to actually using the rules to calculate the number of outcomes, we'll use a calculator or computer.

By the end of this activity, I'm hoping you'll know how to calculate the number of outcomes in an experiment by using either combinations or something I call the "slot method." Let's see if we can derive these counting rules.

10. A local deli advertises a lunch special (sandwich, soup, dessert, and drink) for $5. They offer the following choices:
    - 5 sandwiches: chicken salad, ham, tuna, roast beef, and turkey
    - 4 soups: tomato, chicken noodle, vegetable, broccoli cheese
    - 2 desserts: cookie or cake
    - 5 drinks: water, tea, coffee, milk, juice

    How many different lunch specials are there? See if you can create a tree diagram to display this visually. Then, answer this question using the slot method.

    If you randomly choose a lunch special, what's the probability that it includes a ham sandwich?

11. Suppose you forget to study and you randomly guess on a 10-question true/false quiz. What's the probability that you get a perfect score? What's the probability you get at least one question correct?

> We just derived the **Fundamental Counting Principle (multiplication rule)**
>
> Suppose an experiment consists of a sequence of k choices. If there are n_1 choices in the first stage, n_2 choices in the second stage, and so on, the total number of outcomes in the experiment are:
>
> $$n_1 \times n_2 \times n_3 \times \ldots \times n_k$$

12. In order to create an online bank account, you must select a unique PIN. Your PIN must be 4 characters long and each character may be one of the following:
    - a lower-case letter
    - an upper-case letter
    - a number

    The characters may be repeated. If there are 300 million people in America, could they all have unique PINs?

13. How many PINs could we create if characters cannot be repeated (e.g., you cannot have A1Ag)?

14. My two older brothers and I all have the same initials, “BAT.” According to babynamesworld.parentsconnect.com, there are:
    - 104 commonly used (English) boy names that begin with the letter “B.”
    - 157 commonly used (English) boy names that begin with the letter “A.”

    If my parents chose my first and middle names at random (ensuring I have the initials “BA”), what’s the probability I would have gotten the name Bradley Adam?

15. Let’s continue using the slot method. In how many ways could we arrange the letters ABCD? In this calculation, are we assuming that we are sampling with or without replacement?

16. In how many unique ways can we arrange 10 books on a shelf? If we arrange books randomly, is each outcome equally likely?

We’ve been using the **Factorial Rule**

The number of ways to arrange n objects in a row, called a permutation of the n objects, is equal to

$$n!$$

We can only use this rule when we sample **without replacement**.

17. Suppose we decide to take a class photo. In how many ways could we line up for the photo?

18. Suppose we can only get 5 students in the class photo. How many different line-ups of 5 students could we select?

19. Using each digit only once, how many unique 4-digit numbers are there?

The last two questions are examples of **permutations**.

If we need to fill **r** positions by selecting from **n** different objects, we have ${}_nP_r = \frac{n!}{(n-r)!}$ arrangements

We can only use this rule when we sample **without replacement**.

We can only use this rule when **order does matter** (e.g., ABC is different from ACB)

20. How many ways are there to select a president, vice president, and secretary from 6 people? Is order important?

21. How many ways are there to select 3 committee members from 6 people? How does this differ from the previous question?

The last question was an example of a **combination**.

The number of ways to select r items from n distinct items (where order is not important) is ${}_nC_r = \frac{n!}{(n-r)!r!}$

We can only use this rule when we sample **without replacement**.

We can only use this rule when **order does not matter** (e.g., ABC is the same as ACB, BAC, BCA, CAB, CBA)

22. Will we always have fewer combinations than permutations? Why or why not?

23. How many different 5-card hands can we select from 52 cards? Use your calculator or a computer to get this answer.

24. Suppose we must divide 8 subjects into 2 groups (of 4 subjects each). In how many ways can we do this?

In how many ways can we divide 12 subjects into 3 groups?

Are you sure? Check your answer by randomly assigning 2 people to 2 groups and then 3 people to 3 groups

25. In how many ways can we arrange the letters in the word MISSISSIPPI? Note that we're not distinguishing among the 4 S's, 4 I's, or 2 P's.

    So, another way of phrasing this question would be, in how many ways can we put one M, four I's, four S's, and two P's into an arrangement of 11 letters?

M	I	S	P
	I	S	P
	I	S	
	I	S	

26. Let's turn our attention to some basic probability axioms. Suppose you draw one card from a standard deck of 52 cards (jokers removed). Calculate the following probabilities

P(red card) = ____________________

P(Joker) = ____________________

P(club) = ____________________

P(Ace or Jack) = ____________________

P(not a club) = ____________________

P(ace or club) = ____________________

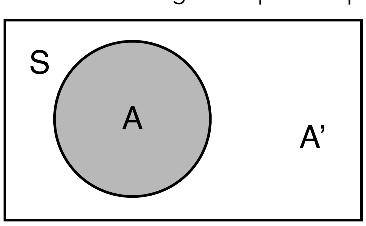
From this simple example, we can generalize basic facts about probabilities. For any event A in sample space S:

- $0 \leq P(A) \leq 1$ Every probability is a number between 0 and 1
- $P(S) = 1$ The probability of the entire sample space is 1.0 (or the sum of all probabilities 1.0)
- $P(A') = 1 - P(A)$ Complement Rule (the complement of A is defined to be "event A does not occur")

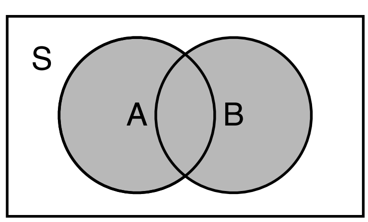
27. We'll find that last rule, the Complement Rule, especially helpful throughout the semester. As a simple example, suppose we know the probability of failing this class is 0.12. What's the probability of not failing this class?

28. If we roll a fair 6-sided die, what's the probability we do not roll a 4?

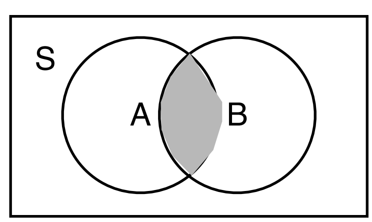
When we calculate probabilities, we'll use counting rules, basic probability rules, and Venn Diagrams.
Shaded areas in a Venn Diagram represent probabilities.



P(A)
Probability of A



P(A or B)
$P(A \cup B)$
A union B
Either A or B occur



P(A and B)
$P(A \cap B)$
A intersect B
Both A and B occur

You'll find Venn Diagrams especially useful when you complete the "practice probability problems"

29. In a statistics class, 10 students are sophomores and 14 are juniors. 11 of the students in the class are engineering majors, but only 3 sophomores are engineering majors. Use a Venn Diagram (or a table) to calculate the following probabilities:

P(sophomore) = ____________________ P(not an engineering major) = ____________________

P(engineering major or sophomore) = ____________________ P(sophomore or junior) = ____________________

Advice on calculating probabilities:

- We will encounter about a dozen probability rules. Most of them will be common sense.
- While these rules can come in handy, don't rely on looking up formulas. Understand the rules.
- When you calculate probabilities, use Venn Diagrams and label all the probabilities you know. Then use the rules.
- All this being said, the General Addition Rule is important and should be memorized and understood.

> Addition Rules: ("or" probabilities)
>
> - General Addition Rule: P(A or B) = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
> - $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Draw Venn Diagrams to display that last rule. Make sure you understand these rules!

30. The Three Stooges starred in 190 movies. The following table displays the movies according to the director and who played the third stooge:

		Director				
		Del Lord	Jules White	Ed Bernds	Other	Total
	Curly	38	33	5	21	97
3rd Stooge	Shemp	1	55	20	1	77
	Joe	0	16	0	0	16
	Total	39	104	25	22	190

- P(Curly is the 3rd stooge) = __________
- P(Del Lord or Jules White is the director) = __________
- P(Curly & Jules White directs) = __________
- P(Curly & Shemp) = __________
- P(Curly or Shemp or Other director) = __________
- P(Joe)′ = __________

Source: http://static.neatorama.com/images/2008-10/three-stooges.jpg

31. Suppose we have 10 IE majors and 10 other majors in this class. I need to choose 4 students at random to fail this course (thus, satisfying my ego). In how many ways could I choose 4 students out of 20? In how many ways could I choose 4 students out of the 10 IE majors? Using the results from these two questions, what is the probability that I randomly select 4 IE majors to fail?