Activity 3: Applications of Counting Rules; Permutation Tests

Situation:	5 engineering majors and 3 math majors are in a statistics class. One day, as the professor walks into the classroom, he happens to notice four \$100 bills on the floor. Rather than splitting the money, the professor decides to give \$100 bills to 2 students. Through a process he claims was random, the professor chose 2 math majors. Do we have reason to believe the professor didn't choose the students at random?

1. Let's begin by examining the sample space in this experiment. In how many ways could the professor choose 2 students out of 8? How can we tell if this is a permutation or combination?

2. Let's list this entire sample space. If I denote engineering majors as **E1-E5** and math majors as **M1-M3**, we have:

E1E2 E1E3 E1E4 E1E5 E2E3 E2E4 E2E5 E3E4 E3E5 E4E5
E1M1 E1M2 E1M3 E2M1 E2M2 E2M3 E3M1 E3M2 E3M3 E4M1 E4M2 E4M3 E5M1 E5M2 E5M3
M1M2 M1M3 M2M3

Were each of these outcomes equally likely to have occurred? What was the likelihood that the professor would choose 2 math majors at random?

3. Using our basic definition of probability, it looks like we just calculated:

$$P\left(2 \text{ math majors}\right)=\frac{\#\text{ of ways of selecting 2 math majors out of 3}}{\#\text{ of ways of selecting 2 students out of 8}}=\frac{\binom{3}{2}}{\binom{8}{2}}=\frac{\frac{3!}{2!(3-2)!}}{\frac{8!}{2!(8-2)!}}=\frac{3}{28}\approx 0.107$$

Based on this result, do we have reason to believe the professor didn't choose the students at random? Explain.

4. Calculate the likelihood of the professor choosing 2 engineering majors at random.

5. Calculate the likelihood of the professor choosing either 2 engineering majors or 2 math majors.

6. Calculate the likelihood of the professor choosing 1 engineering major and 1 math major.

7. Suppose the class had 50 engineering majors and 30 math majors. Further, suppose the professor claimed to have randomly chosen 20 math majors to receive money. What's the probability of this happening if the professor chooses students at random?

Situation:	Suppose I give you a test with 100 questions. You know the answer to 80 of the questions, but you can't even guess the answers to the other 20 questions. Since grading all 100 questions would be too time consuming, I decide to randomly select only 4 questions to grade. If you answered all 4 of those questions correctly, I will assume you answered every question correctly and I will give you an A. If you miss at least one of the 4 questions, I will assume you missed lots of questions and will give you an F. What's the probability that you will get an A on this test?

8. I'm going to randomly select 4 questions out of 100 to grade. In how many ways could I do this?

9. In how many ways could I select 4 questions that you answered correctly?

10. Using your answers to the previous two questions, calculate the probability that you answer all 4 randomly chosen questions correctly.

11. I would hate giving students that much of a chance at getting an A, so how can I reduce this probability?

12. Of the 4 questions I choose at random, what's the probability I choose at least one question you cannot answer?

13. Of the 4 questions I choose at random, what's the probability I choose exactly 1 question you cannot answer? What's the probability I choose exactly 2 questions you cannot answer? Write out the full probability model.

In both examples we've dealt with in this activity, we had equally-likely outcomes. We will often be interested in experiments in which the outcomes are not all equally likely. For example, suppose we toss a coin 3 times:

Sample space = 8 possible outcomes: HHH HHT HTH THH HTT THT TTH TTT
Each outcome is equally likely, so P(HHH) = P(HHT) = ... = P(TTT) = 1/8 = 0.125

Rather than the outcome of each coin toss, suppose we're interested in the **number of heads** we get in 3 tosses.

Sample space = 4 possible outcomes: 0 heads 1 head 2 heads 3 heads
Each outcome is NOT equally likely. The probabilities do NOT all equal 0.25.

P(0 heads) = _______ P(1 head) = _______ P(2 heads) = _______ P(3 heads) = _______

Situation:	You are the world's laziest nurse. One night, 4 mothers with the last names of Anderson, Boyd, Carter, and Dillon give birth to baby boys. Each mother gives her child a first name alliterative to his last: Andy Anderson, Bobby Boyd, Chris Carter, and Dennis Dillon. You're too lazy to keep track of which baby belongs to which mother, so you decide to return the babies to their mothers completely at random. Our goal will be to develop a probability model for this scenario. What's the probability you return 0, 1, 2, 3, or 4 babies to the correct mothers?

14. Before we try to enumerate the outcomes of this experiment, let's run a simulation.
    a. Take 4 evenly-sized scraps of paper and write A on the 1st paper, B on the 2nd, C on the 3rd, and D on the 4th.
    b. Shuffle the papers in your hand and randomly deal them to the 4 "mothers"
    c. Count how many babies you returned to the correct mother
    d. Record this result in one of the blanks below
    e. Repeat this process 4 more times to fill-in all the blanks

Trial #	**1**	**2**	**3**	**4**	**5**
# of correct matches:	_______	_______	_______	_______	_______

Now, let's aggregate the results from everyone in the class to complete the following table:

# of correct matches:	**0**	**1**	**2**	**3**	**4**	**Total**
# of trials in which this happened:	_______	_______	_______	_______	_______	_______
proportion of trials:	_______	_______	_______	_______	_______	_______

15. We've just estimated our probability model through a simulation. Obviously, running simulation is time-consuming, but there's no reason why we couldn't have a computer run the simulation for us: rossmanchance.com/applets/randomBabies/Babies.html

16. Instead of simulating the experiment, we can try to enumerate all possible outcomes to find the probability model. Let's list all possible ways of returning 4 babies to 4 mothers at random. How can we calculate the total number of possible outcomes?

(4) ABCD	(2) BACD	(1) CABD	(0) DABC
(2) ABDC	(0) BADC	(0) CADB	(1) DACB
(2) ACBD	(0) BADC	(0) CADB	(1) DACB
(2) ACBD	(1) BCAD	(2) CBAD	(1) DBAC
(1) ACDB	(0) BCDA	(1) CBDA	(2) DBCA
(1) ADBC	(0) BDAC	(0) CDAB	(0) DCAB
(2) ADCB	(1) BDCA	(0) CDBA	(0) DCBA

From this sample space, write out the probability model for this scenario

17. Four students, who had received A's in statistics all semester, stayed out late the night before a test. Waking up late, they did not make it to class to take the test. Their excuse to the professor was that they had a flat tire (and, therefore, they should be allowed to retake the test). The professor agreed, wrote out a test, and sent the four students into separate rooms to take it. The first question, written on the front of the test, was worth 5 points. All four students answered this question easily. When they flipped to the back of the test, they saw the following item (worth 95 points): **Which tire was flat?**

    What's the probability that all four students choose the same tire?

Those were all examples of how we can use counting rules to estimate probabilities. Now, let's see how we can use counting rules to analyze data using randomization/permutation tests.

Situation: Are yawns contagious? Conventional wisdom says yes: when you see someone else yawn, you're prone to feel sleepy and yawn yourself. How many times have you caught yourself in this situation, or noticed it in someone else? But will this hypothesis withstand a scientific test? Will data support this claim?

The MythBusters investigated this issue by using a two-way mirror and a hidden camera. 50 subjects sat in a booth, accompanied only by an experimental attendee. For some of the subjects, the attendee yawned (planting a yawn "seed"), while for other subjects the attendee did not yawn. The researchers decided in advance, with a random mechanism, which subjects would receive the yawn seed and which would not. As time passed, the researchers watched to see which subjects yawned. They found that 11 of 34 subjects who had been given a yawn seed actually yawned themselves, compared with 3 of 16 subjects who had not been given a yawn seed. These data are summarized in the following 2×2 table:

	Yawn seed planted	Yawn seed not planted	Total
Subject yawned	11	3	14
Subject did not yawn	23	13	36
Total	34	16	50

Do the data appear to support the claim that yawns are contagious?

18. We have a sample of data from 50 subjects. Does this data appear to support the claim that yawns are contagious? Why can't we simply look at this data and conclude that yes, yawns are contagious?

19. Let's assume yawning is NOT contagious. In that case, we could argue that the 14 subjects who yawned in this study would have yawned regardless of whether they had the yawn seed or not.

    If this is true – if yawning is NOT contagious – then how likely were we to randomly assign 11 or more of those 14 yawners into the seed group? In other words, if yawning is not contagious, what's the likelihood that we would have observed our results or something even more extreme?

    To estimate this likelihood, we could conduct a physical simulation:
    - Take 50 pieces of paper and write YAWN on 14 of them (to represent the 14 yawners of our 50 subjects)
    - Shuffle and blindly choose 34 of the papers (representing the 34 subjects assigned to the SEED group)
    - The 16 papers you do not choose represent the 16 subjects assigned to the NO SEED group
    - Record the number of YAWN papers assigned to the SEED group
    - Repeat this process a couple thousand times
    - See how may of those simulations resulted in 11 or more YAWNers being assigned to the SEED group.

We could also speed this up on a computer:

- Go to: http://www.rossmanchance.com/applets/ChiSqShuffle.html?yawning=1
- Uncheck "animate" (to speed things up) and click RANDOMIZE. This runs one simulation, recording the number of yawners randomly assigned to the seed group.
- Run 999 more replications.
- Click APPROX P-VALUE to determine the likelihood of observing 11 or more YAWNers in the SEED group

Go ahead and do this. Record your p-value and briefly write out any conclusions you can make.

20. Let's see if we can calculate an exact p-value from this study. To do this, consider that we have:

    50 total subjects, from which we want to choose 34 (to be assigned to the SEED group)
    14 subjects are yawners
    36 subjects are not yawners

    We want to calculate the probability of choosing 10 or more yawners when we select 34 subjects.

    P(11 yawners from our 34 selected subjects) = _______________

    P(12 yawners from our 34 selected subjects) = _______________

    P(13 yawners from our 34 selected subjects) = _______________

    P(14 yawners from our 34 selected subjects) = _______________

    We could also calculate this using http://stattrek.com/online-calculator/hypergeometric.aspx

Situation:	A company has developed a new drug they believe makes people run faster. To test this claim, they randomly sampled 8 individuals. Four of the subjects were randomly assigned to receive the drug; the other four subjects received a placebo. The researchers then had these eight people run a race. They observed the following results: Drug Group: Finished in 1st, 2nd, 4th, and 5th Placebo Group: Finished in 3rd, 6th, 7th, and 8th place Can we conclude the drug does make people run faster?

21. Why were subjects randomly selected for this study? Why were they randomly assigned to treatment groups? What's the difference between an observational and experimental study?

22. What are some possible reasons why the drug group performed better than the placebo group?

23. Write out the null and alternate hypotheses. Remember that the null hypothesis states that nothing interesting happens (the treatment did not impact the results.

    Null hypothesis:

    Alternate hypothesis:

Based on the results from the race, did the drug group outperform the placebo group? How did you determine this? Restate the hypotheses in mathematical notation.

> As we learned in Activity 1, we will work under the assumption that the null hypothesis is true.
>
> The null hypothesis states that the drug had no impact on the results of the study. If this is true, then the performance of the runners would not have changed if they had been randomly assigned to either group.
>
> For example, we would expect the fastest runner to still finish in first place if he had taken the drug or placebo.
>
> Let's pretend that we're able to go back in time and randomly assign the subjects to groups (again). Since we're assigning subjects randomly, we'd expect that some of the subjects would be assigned to different groups. We are still assuming, however, that the group assignments have no impact on their performance.

24. The following tables show 4 randomizations (different ways subjects could have been randomly assigned to treatment groups).

Randomization 1

Drug	Placebo
1	
2	
	3
4	
5	
	6
	7
	8
Sum = 12	Sum = 24

Randomization 2

Drug	Placebo
	1
2	
3	
4	
	5
	6
	7
8	
Sum = 17	Sum = 19

Randomization 3

Drug	Placebo
1	
2	
3	
4	
	5
	6
	7
	8
Sum = 10	Sum = 26

Randomization 4

Drug	Placebo
1	
	2
	3
	4
	5
6	
7	
8	
Sum = 22	Sum = 14

In which of these randomizations would you conclude that the drug group outperformed the placebo group?

25. This study randomly assigned 8 subjects to 2 groups. How many different ways could we randomly assign 8 subjects into 2 groups? Is each one of these randomizations equally likely to occur?

> We observed an extreme result in which the drug group outperformed the placebo group.
>
> Since we can calculate the total number of possible randomizations and we know each randomization is equally likely to occur, we can calculate the probability of observing results as (or more) extreme as we did.
>
> In other words, we're assuming the drug has no effect on performance. If that's true, we're going to calculate the likelihood of observing our results (the drug group finishing 1st, 2nd, 4th, and 5th place, or better).

To do this, we could list every possible randomization and find those randomizations that yield a sum of 12 or less for the drug group. Obviously, this would be rather time-consuming. An alternate method is to try to list out only the most extreme randomizations that yield a sum of 12 or less for the drug group.

The rows in the following table show possible randomizations. In each row, write 4 X's to indicate which subjects could have been assigned to the drug group. In the final column, calculate the sum of their ranks (places finished). The first two rows are completed.

	1st	2nd	3rd	4th	5th	6th	7th	8th	Sum
Randomization 1	X	X	X	X					10
Randomization 2	X	X	X		X				11
Randomization 3									
Randomization 4									
Randomization 5									
(continue as necessary)									

26. Assuming the drug has no effect on performance, what was the likelihood of observing results as extreme (or more extreme) as the results we did obtain?

27. Would you say our results were likely to have happened if, in fact, the drug does not impact performance? What can you conclude from this study?

Situation: I once missed 11 straight free throws over the course of a few basketball games. Suppose a shooting coach called me and claimed that shooting free throws underhanded is better than shooting them the traditional way. To demonstrate this, this coach randomly selected six subjects from the street. He had 3 subjects shoot 15 free throws underhanded and the other 3 subjects shoot 15 free throws the traditional way.

Here are the number of free throws made by each subject:

Underhand Group: 10 14 8 (total = 32)

Traditional Group: 6 12 9 (total = 27)

From these results, can we conclude that the underhand method is better than the traditional method?

28. State the null and alternate hypotheses for this study. Which hypothesis appears to be correct, based on our observed data?

Null hypothesis:

Alternate hypothesis:

Before we run our test and make our conclusion, let's look at the possible decisions we can make. In reality, one of two things can be true: either the underhand method is better or it isn't. We also will make one of two conclusions: we will decide the underhand method is better or it isn't.

The following table displays the possible realities and decisions. Which of the cells are errors? Which are correct decisions? Describe the results from each cell in the table.

		Reality: Null hypothesis is true	Reality: Null hypothesis is false
Decision	Null hypothesis is true		
	Null hypothesis is false (We reject the null)		

If the null hypothesis is true, the underhand method is not any better than the traditional method. Assuming this is true, the subjects would have made the same number of free throws regardless of which group they were assigned to. For example, if the first subject were assigned to the traditional group, he still would have made 10 free throws.

29. How many different ways could we have randomly assigned six subjects into two treatment groups?

30. Using the following table of randomizations, determine the likelihood of observing results as or more extreme that what we did observe. What conclusions can you make?

Randomization	**6**	**8**	**9**	**10**	**12**	**14**	**Sum**
1	X	X	X				23
2	X	X		X			24
3	X	X			X		26
4	X	X				X	28
5	X		X	X			25
6	X		X		X		27
7	X		X			X	29
8	X			X	X		28
9	X			X		X	30
10	X				X	X	32
11		X	X	X			27
12		X	X		X		29
13		X	X			X	31
14		X		X	X		30
15		X		X		X	32
16		X			X	X	34
17			X	X	X		31
18			X	X		X	33
19			X		X	X	35
20				X	X	X	36