Due to the popularity of MATH 300 and MATH 301, SAU needs to hire additional faculty to teach statistics courses. Suppose 20 Ph.D. statisticians apply for these faculty positions.

If 5 random applicants are chosen for employment, what's the probability that the 5 best statisticians are selected?

Given information: 20 applicants; 5 best applicants; we're selecting 5 at random

We can think of our applicants as belonging to two mutually exclusive groups: the 5 best applicants and the 15 remaining applicants.

When we're randomly selecting objects (applicants) without replacement from two mutually exclusive groups, we can use the hypergeometric distribution to calculate the probability of selecting a certain number of objects from each group.

P(5 best applicants and 10 other applicants) =

$$\frac{\binom{\text{out of 5 best}}{\text{choose 5}}\binom{\text{out of 15 remaining}}{\text{choose 0}}}{\binom{\text{out of 20 total applicants}}{\text{choose 5 total}}} =$$

$$\frac{\binom{5}{5}\binom{15}{0}}{\binom{20}{5}} = \frac{\left(\frac{5!}{(5-5)!5!}\right)\left(\frac{15!}{(15-0)!0!}\right)}{\left(\frac{20!}{(20-5)!5!}\right)} = \frac{(1)(1)}{15504} = 0.0000645$$

If 10 random applicants are chosen for employment, what's the probability that the 5 best statisticians are selected?

$$\frac{\binom{\text{out of 5 best}}{\text{choose 5}}\binom{\text{out of 15 remaining}}{\text{choose 5}}}{\binom{\text{out of 20 total applicants}}{\text{choose 10 total}}} =$$

$$\frac{\binom{5}{5}\binom{15}{5}}{\binom{20}{10}} = \frac{\left(\frac{5!}{(5-5)!5!}\right)\left(\frac{15!}{(15-5)!5!}\right)}{\left(\frac{20!}{(20-10)!10!}\right)} = \frac{(1)(3003)}{(184756)} = 0.01625$$

Suppose we REALLY wanted to hire the 5 best statisticians, no matter how much it cost. If we randomly hire applicants from the pool of 20, how many would we need to hire in order to have a 50% chance of hiring all 5 of the best statisticians?

We already know we have a 0.00645% chance of hiring the 5 best if we only hire 5 total applicants. We also know we have a 1.625% chance of hiring the 5 best if we hire 10 total.

While we could solve this algebraically (with the following formula), it seems easier to just calculate the probability for each possible situation.

$$0.5 \geq \frac{\binom{5}{5}\binom{15}{x}}{\binom{20}{5+x}} = \frac{\left(\frac{5!}{(5-5)!5!}\right)\left(\frac{15!}{(15-x)!x!}\right)}{\left(\frac{20!}{(20-5+x)!(5+x)!}\right)} = \ldots$$

Number of applicants hired	Probability of selecting the 5 best
5	.0000645
6	.000387
7	.00135
8	.0036
9	.0081
10	.01625
11	.0298
12	.0511
13	.0830
14	.1291
15	.1937
16	.2817
17	.3991
18	**.5526 (over 50%)**
19	.7500
20	1.0

If we hire 5 applicants, what's the probability we'll hire at least 2 of the 5 best applicants?

The probability of choosing at least 2 out of the 5 best means we'll have to separately calculate the probabilities of choosing exactly 2, 3, 4, and 5 of the best applicants:

$$P(X \geq 2 \text{ best}) = P(x{=}2) + P(x{=}3) + P(x{=}4) + P(x{=}5)$$

A faster way to calculate the answer would be to use the complement rule:

$$P(X \geq 2 \text{ best}) = 1 - P(X \leq 1) = 1 - \left[P(X=1) + P(X=0)\right]$$

$$1 - \left[\frac{\binom{5}{1}\binom{15}{4}}{\binom{20}{5}} + \frac{\binom{5}{0}\binom{15}{5}}{\binom{20}{5}}\right] = 1 - [.44 + .19] = 0.37$$