The Poisson Probability Distribution (Binomial in disguise)

Suppose we want to find the probability distribution of the number of automobile accidents at a particular intersection during a one-week period. How could we derive this probability distribution?

Think of the one-week time period as being split into n subintervals. In fact, think of the subintervals as being so small that *at most one accident could occur in each subinterval.* If we let p represent the probability of an accident in any subinterval, we have:

P(no accidents occur in a subinterval) = 1 – p
P(one accident occurs in a subinterval) = p
P(>1 one accident occurs in a subinterval) = 0 (the subinterval is too small to allow this)

We can now think of the total number of accidents during the week as the *total number of subintervals that contain one accident*. If the occurrence of accidents from interval to interval can be assumed to be independent, we now have:

An independent number of trials (subintervals) with two possible outcomes (accident/no)
A constant probability of an accident occurring (p)
A question asking for the probability of observing a certain number of accidents in a certain number of trials

These conditions tell us that we're dealing with a binomial distribution.

How do we go about dividing the one-week period into small subintervals (so small that there is zero probability of observing more than one accident in a subinterval)? How many subintervals do we need? I don't know. It seems reasonable, though, that as we divide the week into a greater number of n subintervals, the probability p of one accident in one of these shorter subintervals will decrease.

Let's let $\lambda = np$ and let's take the limit of the binomial distribution as $n \to \infty$:

Recall the formula for binomial probabilities: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

If we let the number of trials approach infinity, we have:

$$\lim_{n\to\infty} \binom{n}{x} p^x (1-p)^{n-x} = \lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-x)!}{x!(n-x)!} p^x (1-p)^{n-x} = \lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-x+1)!}{x!} p^x (1-p)^{n-x}$$

Since $\lambda = np$, we can write: $\displaystyle \lim_{n\to\infty} \frac{n(n-1)(n-2)\ldots(n-x+1)!}{x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$

We can rearrange terms: $\displaystyle \lim_{n\to\infty} \left(\frac{\lambda^x}{x!}\right)\left(1-\frac{\lambda}{n}\right)^n \frac{n(n-1)(n-2)\ldots(n-x+1)}{n^x}\left(1-\frac{\lambda}{n}\right)^{-x}$

$$= \left(\frac{\lambda^x}{x!}\right) \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x} \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)$$

$$= \left(\frac{\lambda^x}{x!}\right) \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n (1)$$

How do we evaluate this limit? Since we have an indeterminate form, we need L'Hopital's Rule:

$$y = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$$

$$\ln(y) = \lim_{n\to\infty} n\ln\left(1-\frac{\lambda}{n}\right)$$

$$\ln(y) = \lim_{n\to\infty} \ln\left(\frac{1-\frac{\lambda}{n}}{\frac{1}{n}}\right)$$

$$\ln(y) = \lim_{n\to\infty}\left(\frac{\frac{\frac{\lambda}{n^2}}{1-\frac{\lambda}{n}}}{\frac{-1}{n^2}}\right)$$

$$\ln(y) = \lim_{n\to\infty} \ln\left(\frac{-\lambda}{1-\frac{\lambda}{n}}\right) = -\lambda$$

so $$y = e^{-\lambda} = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$$

So we now know that:

$$\left(\frac{\lambda^x}{x!}\right)\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = \left(\frac{\lambda^x}{x!}\right)e^{-\lambda}$$

Thus, our probability of interest is:

$$P(x) = \frac{\lambda^x e^{-x}}{x!}$$

That's a binomial probability as the number of trials approaches infinity. In our situation, it's the probability of observing x automobile accidents per week.