Continuous Random Variable Practice:

Complete this after Activity #9.

Situation: The probability that a battery will hold a charge for X years is modeled by $f(x) = \frac{2}{(x+1)^3}$.

1. Show that this is, in fact, a valid probability function.

2. Find the probability that a randomly selected battery lasts between 2 and 5 years.

3. Find the probability that a randomly selected battery lasts more than 5 years.

4. Find the probability that a randomly selected battery lasts *5 years or more*.

5. Find the probability that a randomly selected battery lasts exactly 5 years.

6. Write out the cumulative distribution function (cdf) and use it to calculate the probability of a battery lasting between 2 and 5 years.

7. Find the median (50^{th} percentile) number of years a battery lasts.

8. Find the 90th percentile of the number of years a battery lasts.

Answers:

1. Show that this is, in fact, a valid probability function.

First, I must show that this function is always positive (because all probabilities must be greater than zero).

The domain of my function is from 0 to infinity. Therefore, x must be a positive number. The function $f(x) = \frac{2}{(x+1)^3}$ is positive for all positive values of X, so this first condition is met. I could also simply graph this function and observe that it is always positive over the domain of interest.

Next, I must show that the area under the curve sums to 1.00.

I need to show $\int_0^\infty \frac{2}{(x+1)^3} = 1.00$.

I can rewrite: $\int_0^\infty \frac{2}{(x+1)^3} = 2\int_0^\infty (x+1)^{-3}$ and then use u-substitution:

Let $u = x+1$ so that $du = dx$. Then I can rewrite: $2\int u^{-3}du$

Then I can find the integral: $2\frac{u^{-2}}{-2} = -u^{-2} = -(x+1)^{-2} = \left.\frac{-1}{(x+1)^2}\right|_0^\infty$

Evaluating the endpoints, I can find the area under the curve:

$$\left.\frac{-1}{(x+1)^2}\right|_0^\infty = \left[\frac{-1}{(\infty+1)^2}\right] - \left[\frac{-1}{(0+1)^2}\right] = \left[\frac{-1}{\infty}\right] - \left[\frac{-1}{1^2}\right] = 0-(-1) = 1.00$$

2. Find the probability that a randomly selected battery lasts between 2 and 5 years.

$$\int_2^5 \frac{2}{(x+1)^3} = \left.\frac{-1}{(x+1)^2}\right|_2^5 = \left[\frac{-1}{(5+1)^2}\right] - \left[\frac{-1}{(2+1)^2}\right] = \left[\frac{-1}{36}\right] - \left[\frac{-1}{9}\right] = 0.0833$$

Or, simply, $F(x) = \frac{-1}{(x+1)^2}$ so that $P(2 < X < 5) = F(5) - F(2) = 0.0833$

3. **Find the probability that a randomly selected battery lasts more than 5 years.**

$$P(X>5)=1-P(X\le 5)=1-\int_0^5 \frac{2}{(x+1)^3}=1-\left[\left.\frac{-1}{(x+1)^2}\right|_0^5\right]=$$

$$\left.\frac{-1}{(x+1)^2}\right|_0^5=\left[\frac{-1}{(5+1)^2}\right]-\left[\frac{-1}{(0+1)^2}\right]=\left[\frac{-1}{36}\right]-\left[\frac{-1}{1}\right]=0.9722$$

So $P(X>5)=1-P(X\le 5)=1-\int_0^5 \frac{2}{(x+1)^3}=1-\left[\left.\frac{-1}{(x+1)^2}\right|_0^5\right]=1-.9722=0.0278$

4. **Find the probability that a randomly selected battery lasts *5 years or more*.**

$P(X>5)=P(X\ge 5)=0.0278$

5. **Find the probability that a randomly selected battery lasts exactly 5 years.**

For a continuous random variable, $P(X=5)=0$

6. **Write out the cumulative distribution function (cdf) and use it to calculate the probability of a battery lasting between 2 and 5 years.**

$F(x)=\frac{-1}{(x+1)^2}$. See the answer to problem #2

7. **Find the median (50th percentile) number of years a battery lasts.**

Solve the following for a: $\int_0^a \frac{2}{(x+1)^3}=0.50$

$$\left.\frac{-1}{(x+1)^2}\right|_0^a=\left[\frac{-1}{(a+1)^2}\right]-\left[\frac{-1}{(0+1)^2}\right]=\left[\frac{-1}{(a+1)^2}\right]-\left[\frac{-1}{1}\right]=0.50 \text{ so } \frac{-1}{(a+1)^2}=-0.50$$

$-1=-0.5(a+1)^2$ so $2=(a+1)^2$ and then $\sqrt{2}=a+1$ so that $\sqrt{2}-1=a$.

The median is approximately 0.414 years

8. **Find the 90th percentile of the number of years a battery lasts.**

Solve the following for a: $\int_0^a \frac{2}{(x+1)^3}=0.90$.

We find that the 90th percentile is approximately 2.162. (Exact value is $\sqrt{10}-1=a$)