Activity #3a: Variances; Chi-Square and F-distributions; hypothesis tests (part 2)

In the last activity, we learned that the sampling distribution of the sample variance is a chi-squared distribution with n – 1 degrees of freedom. We then learned how to construct confidence intervals for population variances. In this activity, we will learn how to compare the variances between two groups.

1) Consider two situations:

   A) A hospital must select between two brands of blood pressure monitors. After calibrating both brands, they test each monitor 21 times. The first brand, a traditional armband/pump device which is relatively inexpensive, had a standard deviation in its measurements of 6.1 (variance of 37.21 units). The second brand, a more expensive but less intrusive automated device, had a standard deviation of 3 units (variance = 9.00 units). The hospital must decide which brand to buy based on the precision of the instruments.

   B) One supplier provides upholstery fabric with an average durability of 74,283 DR and a standard deviation of 4676 DR (variance = 21,864,976). Another supplier provides fabric with a lower average durability of 74,200 DR and a lower standard deviation of 4500 DR (variance = 20,250,000). All measurements are based on a sample of n=61. You must decide which supplier to purchase from based on the variance of their fabric

   When we compared means using an independent samples t-test, our test statistic was based on the difference between those two means (simple subtraction). Calculate the difference in variances for each situation. Which situation appears to have the biggest difference between variances?

2) Based on the differences in variances, we might believe that situation B has a bigger disparity between groups. Instead of a simple difference, try calculating a ratio of the variances in each situation. Express what each ratio represents.

3) Which measure, the difference or the ratio between variances, will provide the best measure of the difference between sample variances? Why?

4) In order to complete a hypothesis test comparing the two variances, we will need to derive a sampling distribution of the ratio between the two variances. We start with the formulas for the unbiased sample variances:

$$s_1^2 = \frac{\sum_{i=1}^{n}\left(x_i - \overline{X}_1\right)^2}{n_1 - 1} \qquad\qquad s_2^2 = \frac{\sum_{i=1}^{n}\left(x_i - \overline{X}_2\right)^2}{n_2 - 1}$$

In the last activity, we learned that a Chi-Square distribution has the form:

$$\chi^2_{n_1-1} \sim \frac{(n_1-1)s_1^2}{\sigma_1^2} \qquad\qquad \chi^2_{n_2-1} \sim \frac{(n_2-1)s_2^2}{\sigma_2^2}$$

Using some simple algebra, we can find the distribution of:

$$\frac{s_1^2}{\sigma_1^2} \sim \frac{\chi^2_{n_1-1}}{n_1-1} \qquad\qquad \frac{s_2^2}{\sigma_2^2} \sim \frac{\chi^2_{n_2-1}}{n_2-1}$$

Now we know the sampling distribution of each sample variance.

We want to know the sampling distribution of the *ratio* of the variances. If we assume the two population variances are equal, we have:

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} = \frac{s_1^2}{s_2^2} \sim \frac{\left(\frac{\chi^2_{n_1-1}}{n_1-1}\right)}{\left(\frac{\chi^2_{n_2-1}}{n_2-1}\right)}$$

We just derived the F-Distribution, named after Sir Ronald A. Fisher, a British mathematician and biologist who is credited with discovering p-values and ANOVA). The ratio of two chi-squares each divided by their degrees of freedom is distributed as an F-distribution with degrees of freedom v_1 and v_2.:

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim \frac{\left(\frac{\chi^2_{n_1-1}}{n_1-1}\right)}{\left(\frac{\chi^2_{n_2-1}}{n_2-1}\right)} \sim F_{v_2}^{v_1} = F_{n_2-1}^{n_1-1}$$

Look at that result. The ratio of two sample variances follows an F-distribution. Thus, if we want to compare two variances, we take their ratio and use an F-distribution.

As with other continuous distributions, when we want to calculate probabilities, we need to find the area under a curve. The F-distribution has the function:

$$F\left(\frac{\chi_1^2/v_1}{\chi_2^2/v_2}\right) = \frac{\Gamma\frac{v_1+v_2}{2}}{\Gamma\frac{v_1}{2}\Gamma\frac{v_2}{2}}\left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}}\left(\frac{\chi_1^2/v_1}{\chi_2^2/v_2}\right)^{\frac{v_1}{2-1}}\left(1+\frac{v_1}{v_2}\left(\frac{\chi_1^2/v_1}{\chi_2^2/v_2}\right)\right)^{\frac{-(v_1-v_2)}{2}}$$

To find areas under the F-distribution, we'll use a table or a computer.

5) Let's get some practice with our F-distribution table. The top of the table shows an example of an F-distribution. You can see that it's positively skewed, like the chi-square distribution. If you want to see how the distribution changes as its degrees of freedom change, you can try http://www.capdm.com/demos/software/html/capdm/qm/fdist/usage.html

a) Sketch an F-distribution with 20 degrees of freedom in the numerator and 10 degrees of freedom in the denominator. Using the table, find the critical value that cuts-off 0.05 to the right. Label this on your sketch and shade in the 5% area.

b) Now find the critical value that cuts-off 0.10 to the right. Label this on your sketch and shade in the 10% area.

c) Now find the critical value that cuts-off 0.05 on the left side of the curve. Can you find this in your table?

6) We need to learn how to find critical values on the left side of an F-distribution. Let's begin by examining the F-distribution again. By definition, an F-distribution is:

$F_{v_2}^{v_1} \sim \dfrac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} = \dfrac{\chi_1^2/v_1}{\chi_2^2/v_2}$. What would happen if we switched the degrees of freedom. What's the formula for $F_{v_1}^{v_2}$?

Answer: $F_{v_1}^{v_2} \sim \dfrac{s_2^2/\sigma_2^2}{s_1^2/\sigma_1^2} = \dfrac{\chi_2^2/v_2}{\chi_1^2/v_1} = \dfrac{1}{F_{v_2}^{v_1}}$

This gives us the solution to our problem of finding critical values on the left side of an F-distribution. To find critical values on the left side, we simply need to remember:

$$F_{v_1,\ 1\text{-}\alpha}^{v_2} = \frac{1}{F_{v_2,\ \alpha}^{v_1}}$$

7) Given an F-distribution with 20 degrees of freedom in the numerator and 10 degrees of freedom in the denominator, find the critical values that cut-off 5% and 10% to the left. In other words:

$F_{10,\ 0.95}^{20} =$ ______________________ $F_{10,\ 0.90}^{20} =$ ______________________

Hint: Finding critical values on the left side of an F-distribution is a bit of a pain. When you are comparing two variances, always put the larger variance on top to avoid this aggravation.

8) Consider situation A from the first page of this hand-out. Let's conduct an F-test (at a 0.05 significance level) to compare the variances of the two blood pressure monitors.

A hospital must select between two brands of blood pressure monitors. After calibrating both brands of monitors, they test each monitor 21 times. The first brand, a traditional armband/pump device which is relatively inexpensive, had a standard deviation in its measurements of 6.1 (variance of 37.21 units). The second brand, a more expensive but less intrusive automated device, had a standard deviation of 3 units (variance = 9.00 units). The hospital must decide which brand to buy based on the precision of the instruments.

a) Let's first write out our null and alternate hypotheses. Recall that we'll work under the assumption that the null hypothesis is true.

H_0:

H_A:

b) To compare two variances, we calculate an F-statistics by taking their ratio. To simplify the process, we put the larger variance on top. Go ahead and calculate this test statistic. How many degrees of freedom will we have in the numerator and denominator?

$$F_{v_2}^{v_1} = \frac{s^2_{\text{larger}}}{s^2_{\text{smaller}}} =$$

c) This is our test statistic for this particular set of samples of size *n=21.* But what if we could go back in time and repeat this experiment with another set of random samples? What if we could go back in time and run this experiment an infinite number of times, each time with a potentially different random sample? If we calculated a ratio of variances for each of our replications, what would the distribution of those ratios looks like? Sketch this distribution and label its mean.

We know that $F_{v_2}^{v_1} \sim \frac{s_1^2 / \sigma_1^2}{s_2^2 / \sigma_2^2}$, but we only have $\frac{s_1^2}{s_2^2}$. If we assume the null hypothesis is true, then $F_{v_2}^{v_1} \sim \frac{s_1^2}{s_2^2}$

d) Find the critical value for α=0.05 and label it on your sketch above.

e) Make your decision and write out the conclusion for this study.

Complete the following problems and turn them in at the beginning of our next class meeting:

Problem A:

One supplier provides upholstery fabric with an average durability of 74,283 DR and a standard deviation of 4676 DR (variance = 21,864,976). Another supplier provides fabric with a lower average durability of 74,200 DR and a lower standard deviation of 4500 DR (variance = 20,250,000). All measurements are based on a sample of n=61. You must decide which supplier to purchase from based on the variance of their fabric.

Conduct an F-test (at a 0.01 significance level) comparing the variances in durability of the two fabric suppliers. Write out your hypotheses, sketch the sampling distribution, calculate your test statistic, and write out your conclusion.

Problem B:

The English Department has two instructors grade 30 student essays. Each instructor grades each essay on a scale from 0-100. The Department is interested in whether the instructors are grading students similarly.

Once grading was completed, the Department calculated the average essay score from each instructor. They found that the instructors had similar average scores. They also found that the scores given by Instructor A had a variance of 52.3 and the scores given by Instructor B had a variance of 68.9.

Test the claim that the instructors had different variances in grading essays. Conduct an F-test (at a 0.05 significance level). Write out your hypotheses, sketch the sampling distribution, calculate your test statistic, and write out your conclusion.