Activity #8: Logarithmic Functions, Logistic Regression (student learning outcomes listed in syllabus)

1) Recall this scenario from the last activity:

In October of 2011, Forbes magazine noted that Lady Gaga had 15 million followers on Twitter and that the number of followers was increasing by 6% each month

We modeled the growth of her followers over time with the exponential model: $f(t) = 15(1.06)^t$ (t = time in months)

According to this model, how long will it take for Lady Gaga to reach 45 million followers? Estimate this answer graphically.

2) Let's find an exact solution for this question. We want to solve the following equation for *t*: $15(1.06)^t = 45$. Let's isolate our variable, *t*. What would be our first step? Why would we do this?

3) If we do this, we're left with $(1.06)^t = 3$. What can we do to isolate *t*?

A **logarithm** is the inverse of the exponential function.

4) I guess that box doesn't really tell us what a logarithm is; it tells us what it's the inverse of. Let's see if we can figure out what a logarithm is.

What is an inverse function? If g(x) is the inverse of f(x), f(g(x)) = g(f(x)) = ________.

Let's start with the general formula for an exponential function and go through the steps to find its inverse.

$$f(x) = a\left(1+r\right)^x$$

a represents the initial value in an exponential function. To simplify things, let's set *a* = 1. Also, let's set 1 + *r* = b.

$$y = b^x$$

How can we find the inverse of this function?

5) Now that we've found the inverse of an exponential function, what is it? What does the *y* represent?

If x > 0, the logarithm of x ($\log_b x$) is simply the exponent of b that yields x.

In other words, if $y = \log_b(x)$, then $b^y = x$.

6) Let's reinforce this definition with a few examples.

a) Write the following as logarithmic equations: $2^3 = 8$ $5^2 = 25$ $10^{-2} = 0.01$

b) Write the following as exponential equations: $\log_3 9 = 2$ $\log_4 64 = 3$ $\log_{10} 0.1 = -1$

c) Evaluate the following expressions: $\log_6(36)$ $\log_2(0.5)$ $\log_{10}(1000)$ $\log_7(1)$

d) If you're given $10^{2.8} = 630.957$, then $\log(630.957) =$ ___________?

e) $\log_b(b^n) =$ _________

6) Recall that we set $b = 1 + r$, where r is our growth rate. What's the smallest possible value for r? Since that growth rate could be anything higher than that value, we could have logarithms with any positive base. In this class, we will work almost exclusively with base 10 (b = 10).

Use your calculator to verify the answers to the following logarithms:

$\log(1000) =$

$\log(100) =$

$\log(10) =$

$\log(1) =$

$\log(0.1) =$

$\log(0.01) =$

$\log(-10) =$

$\log(10^x) =$

7) Estimate the following:

$\log(250) =$

$\log(5000) =$

8) Solve for x: $480(10^{0.06x}) = 1320$

9) Let's (finally) solve the first question on this page. When will Lady Gaga have 45 million followers on Twitter?

$$15(1.06)^t = 45$$

10) Solve the following:

$$10^{.5x} = 10{,}000$$

$$\log(10^{banana}) = \_\_\_\_\_\_\_\_\_\_$$

$$\log(3x + 2) + 1 = 0$$

$$4(1.171^x) = 7(1.088^x)$$

11) If the population of the U.S. is 315 million with an annual growth rate of 0.7%, how many months will it take before Lady Gaga followers outnumber people in the U.S.?

12) In the last activity, we assumed your body metabolizes 16% of the caffeine in your system each hour. If you drink two cups of coffee at 8:00 am (containing a total of 200 mg of caffeine), the amount of caffeine that remains in your body after t hours is modeled by: $C(t) = 200(0.84)^t$. When will you have exactly 25 mg of caffeine in your body?

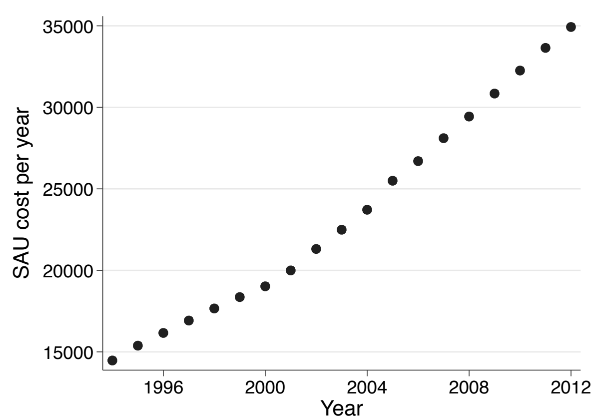
13) Bismuth-210 is an isotope that decays by about 13% each day. What is the half-life of Bismuth-210?

14) Cesium-137 has a half-life of about 30 years. Find the annual decay rate.

15) Carbon-14, with a half-life of 5730 years, is a radioactive isotope that is present in organic materials, and is commonly used for dating historical artifacts. If a bone fragment is found that contains 20% of its original carbon-14, how old is the bone?

16) In activity #1, we saw the cost of attending St. Ambrose (tuition, room, and board) from 1994-2012:

Year	Cost	Year	Cost
1994	14470	2004	23716
1995	15380	2005	25496
1996	16160	2006	26700
1997	16920	2007	28106
1998	17660	2008	29434
1999	18360	2009	30844
2000	19020	2010	32255
2001	19994	2011	33644
2002	21310	2012	34926
2003	22490		



Source: http://www.sau.edu/Institutional_Research/Reports.html

a) Use a calculator to fit a linear model to this dataset. Write out the formula for the linear function and interpret the slope.
b) Use a calculator to fit an exponential model. Interpret the parameters of this model.
c) According to the linear model, when will it cost $50,000 to attend SAU?
d) According to the exponential model, when will it cost $50,000 to attend SAU?

17) On your calculator, graph y = log(x) and y = 10^x. Sketch the graphs below and explain how the graph demonstrates the functions are inverses of each other. Identify the domain and range of each function.

18) In the last activity, we encountered *e*, the natural base. On your calculator, graph y = e^x and y = ln(x). The graph demonstrates that the natural logarithm is the inverse of the exponential function with base *e*. To further demonstrate this, evaluate the following on your calculator:

$$\ln(e^3) =$$

$$\ln(e^{-5}) =$$

$$e^{\ln(15)} =$$

$$e^{\ln(27)} =$$

19) Instead of writing this activity, I'd rather be in Disney World. I went to the Disney website and found how much it costs for a ticket. I found that if you buy a ticket for more days, you pay less per day. The cost for a ticket with the *Park Hopper®* and *Water Park Fun & More* options are displayed below.

Days	Total cost
1	$168
2	255
3	321
4	335
5	347
6	357
7	367
8	377
9	387
10	397

Enter this data into your calculator and create a scatterplot. Sketch the graph below.

What function appears to model this data best: linear, exponential, or logarithmic?

Fit a logarithmic function to this data by selecting the LnReg option.

Write the formula for this logarithmic function below. Record the value for r^2.

Graph this function on top of your scatterplot. Does it appear as though this function fits the data well?

According to our model, how many days would we need for a total cost of $500?

Source: http://tickets.disney.go.com/

20) The following table displays the number of municipal governments as a function of time. Fit a logarithmic function to this data, write out the formula, and comment on how well the function fits the data. According to our model, when would we have 20,000 municipal governments?

Years since 1965	# of municipal governments
2	18,048
7	18,517
12	18,862
17	19,076
22	19,200
27	19,279
32	19,372
37	19,429

21) One important fact about exponential functions is that growth, ultimately, becomes very rapid. Consider a population that is growing 2% each year. At first, growth seems pretty slow; but it means that the population will double every 35 years.

It seems unrealistic for a population to grow exponentially forever. In most situations, there are factors that will limit growth (such as available land, natural resources, or food supplies).

Consider the example we looked at in question #20. It's not possible for the cost of attending SAU to go up exponentially forever... right?

In situations where we want to model growth that will eventually slow down over time, we can use **logistic functions**.

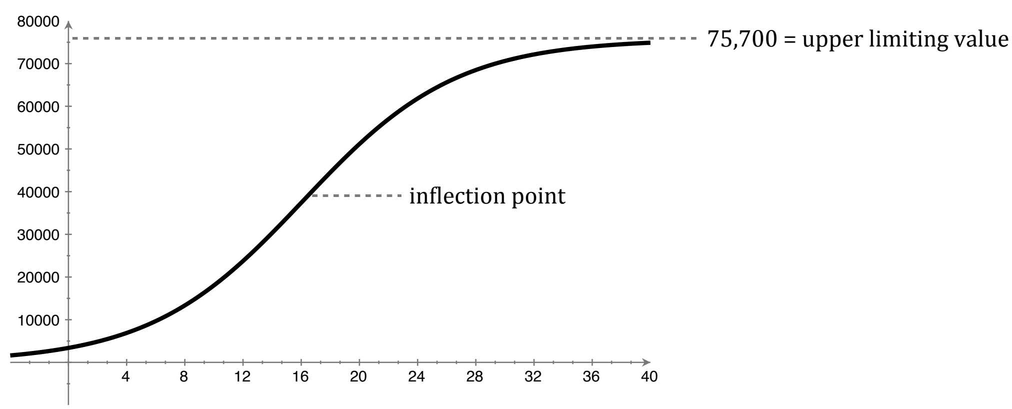
A **logistic function** modeling growth over time has the form:

$$f(t) = \frac{C}{1 + Ae^{-Bt}}$$

where C = the upper limiting value of the function

The Centers for Disease Control (CDC) monitor the number of children age 0-4 who visit their sentinel providers with flu-like symptoms. The following logistic function models the number of children who visited each week during the 2006-07 flu season:

$$f(t) = \frac{75,700}{1 + 21.31e^{-0.1897t}}$$



22) What is an inflection point? Why would it be important in this scenario?

23) The infant mortality rate (deaths per 1,000 live births) has been falling since 1950. Enter the following data into your calculator and find the logistic function that best fits this data. Sketch the scatterplot and function below. Write out the function and comment on how well the model fits the data.

Years since 1950	Infant mortality rate
0	29.2
10	26.0
20	20.0
30	12.6
35	10.6
40	9.2
45	7.6
50	6.9

24) The following table displays the number of European and American countries issuing postage stamps from 1840 to 1880. Model this data using a logistic function.

Year	Counties
1840	1
1845	3
1850	9
1855	16
1860	24
1865	30
1870	34
1875	36
1880	37