Activity #7: Binomial Distribution, Binomial Test, Sign Test (with a chance to win 1,000,000 extra credit points)

Situation: Suppose I give you a multiple-choice quiz where each question asks for the middle name of one of my family members.

Example question: What is my mom's middle name?

a) Elizabeth
b) Lee
c) Marie
d) None of the above

In order to pass the quiz, you need to answer at least half of the questions correctly.

1. Assuming you're going to randomly select your answers to each question, are you more likely to pass a shorter quiz with 4 questions or a longer quiz with 10 questions? Explain.

2. Let's take both the shorter and longer versions of this quiz. Since you're going to guess the answers anyway, I won't even bother reading the questions. Instead, I'll have a computer randomly select the answer to each question. Write your answers (A, B, C, or D) to each question in the following table

	1	2	3	4	5	6	7	8	9	10
Shorter Quiz										
Longer Quiz										

Note: If you get a perfect score on the longer quiz, you will win **1,000,000** extra credit points.

3. Record the number of students who passed the 4-question quiz: __________ the 10-question quiz: __________.
Which quiz seems easier to pass? Why?

4. Suppose we're interested in the number of items you answer correctly. Is this a discrete or continuous random variable? Is each outcome equally likely?

5. Suppose we're interested in your score on each item. Is this a discrete or continuous random variable? What's the probability you answer any single item correctly or incorrectly?

6. Let X = a student's score on any single item (so X can equal 0 or 1). Calculate the following:

   E(x) = ____________________

   Var(x) = ____________________

A **Bernoulli** random variable can take one of two possible values: 0/1, alive/dead, success/fail.

The probability of success for a Bernoulli random variable is defined as: $P(X = 1) = p$

7. Let's generalize our calculations. If we have any Bernoulli random variable with probability p, what are the expected value and variance?

   E(x) = ____________________

   Var(x) = ____________________

8. If the questions are independent -- if P(2nd question correct | 1st question correct) = P(2nd question correct) -- we can think of the quiz as a series of independent Bernoulli trials in which $p = 0.25$.

   Is it reasonable to assume your scores on each item are independent on this quiz? Are questions on a real class quiz independent of one another? What does independence mean?

9. On the 4-question quiz, we could answer all questions correctly (1111) or all questions incorrectly (0000). We could also answer one question correctly (1000, 0100, 0010, or 0001), or two, or three. How many different outcomes could we get from the 4-question quiz? Write out the entire sample space.

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10. Look at the sample space you just listed. Are each of those outcomes equally likely? In other words, does P(1111) = P(0000) = P(1010) = ... ?

11. Let's calculate the probability of getting a perfect score on the 4-question quiz.

Why can't we just look at the sample space and calculate the probability as 1/16 = 0.0625?

Number of outcomes in which you get a perfect score: ___________________

Probability of getting a perfect score = P(1111) = P(X = 4): ____

12. Let's calculate the probability of getting a score of zero on the 4-question quiz.

Number of outcomes in which you get no items correct: ___________________

Probability of getting a perfect score = P(0000) = P(X = 0): ____

13. Let's calculate the probability of getting a score of 1 on the 4-question quiz.

Number of outcomes in which you get a total score of 1 out of 4: ___________________

P(1000): ___________________

Probability of getting a perfect score = P(X = 1) = ____

14. Let's calculate the probability of getting a score of 2 on the 4-question quiz.

Number of outcomes in which you get a total score of 2 out of 4: ___________________

How can we calculate the number of outcomes without looking at the sample space?

P(1100): ___________________

Probability of getting a perfect score = P(X = 2) = ____

15. Let's generalize our calculations. Find a formula that will allow you to calculate P(X = x) when we have a series of independent Bernoulli trials with probability of success, p.

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Binomial Distribution:

If we have a series of n independent Bernoulli trials ($X_1, X_2, \ldots, X_n$) with each trial having constant probability of success p, then the random variable X = X1 + X2 + … + Xn has a Binomial distribution.

Conditions: A series of *n* independent trials
Each trial has two possible outcomes (0/1, success/fail)
The probability of success, p, remains constant across all trials
We're interested in X = the number of successes in *n* trials

Expected value: E[x] = np   Variance: Var[x] = np(1-p)

Probability Mass Function: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \text{binompdf(n, p, x)}$

Cumulative Distribution: P(X ≤ x) = binomcdf(n, p, x)

16. Use your calculator or a computer to complete the probability model for the 4-question quiz.

x	0	1	2	3	4
P(x)	0.316	0.422			0.004

17. Use the probability model to calculate the probability that a randomly selected student will pass the quiz. Use your calculator to verify your answer.

18. What's the expected value for your score on the 4-item quiz. What does this value represent?

19. Use your calculator to find the probability that a student passes the 10-question quiz. Was it more difficult to pass the 4-question or 10-question quiz? Why?

20. What's the probability of rolling three or more 6's in five tosses of a die? Verify the conditions for a binomial distribution have been satisfied.

21. Suppose 10% of the population is left-handed. How many left-handed students do we have in this class? What's the likelihood of seeing at least this many left-handed students in a class of this size? Again, verify the conditions have been met.

22. At the beginning of this activity, you each had a chance to win 1,000,000 extra credit points by getting a perfect score on the 10-question quiz.

    a) Calculate the probability of a student getting the 1,000,000 extra credit points.

    b) Suppose 30 students take this quiz each year and I teach this course for 30 years. What's the probability that at least one student over those 30 years will win the 1,000,000 extra credit points?.

23. During the 2010-11 school year, almost 1.7 million students took the ACT. 5% of those students received accommodations. Suppose we sample 100 of these students at random. What's the probability we'll select at least one student who received accommodations? Have the conditions for a binomial distribution been met?

Source: http://blogs.edweek.org/edweek/speced/2012/05/more_students_receiving_accomm.html

Derivation of the Expected Value of the Binomial Distribution:

Recall that for any discrete random variable: $E[x] = \sum_{i=1}^{n} x_i P\left(x_i\right)$

For a binomial random variable with probability of success *p*: $P(x) = \binom{n}{x} p^x \left(1-p\right)^{n-x}$

$$E[x] = \sum xP\left(x\right) = \sum x \binom{n}{x} p^x \left(1-p\right)^{n-x}$$

$$= \sum x \frac{n!}{x!(n-x)!} p^x \left(1-p\right)^{n-x}$$

$$= \sum x \frac{n(n-1)!}{x(x-1)!(n-x)!} pp^{x-1} \left(1-p\right)^{n-x}$$

$$= \sum np \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} \left(1-p\right)^{n-x}$$

$$= np\sum \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} \left(1-p\right)^{n-x} = np\sum \frac{(n-1)!}{(x-1)!\left[(n-1)-(x-1)\right]!} p^{x-1} \left(1-p\right)^{n-x}$$

Look at what is to the right of the summation. When we sum all of that, what do we get? To make things more clear, let's let a = (n-1) and b = (x-1). We then have:

$$np\sum \frac{a!}{b!\left[a-b\right]!} p^a \left(1-p\right)^b$$

$$= np\sum \binom{a}{b} p^a \left(1-p\right)^b = np\left[\text{the sum of the probabilities of all possible outcomes in a binomial distribution}\right]$$

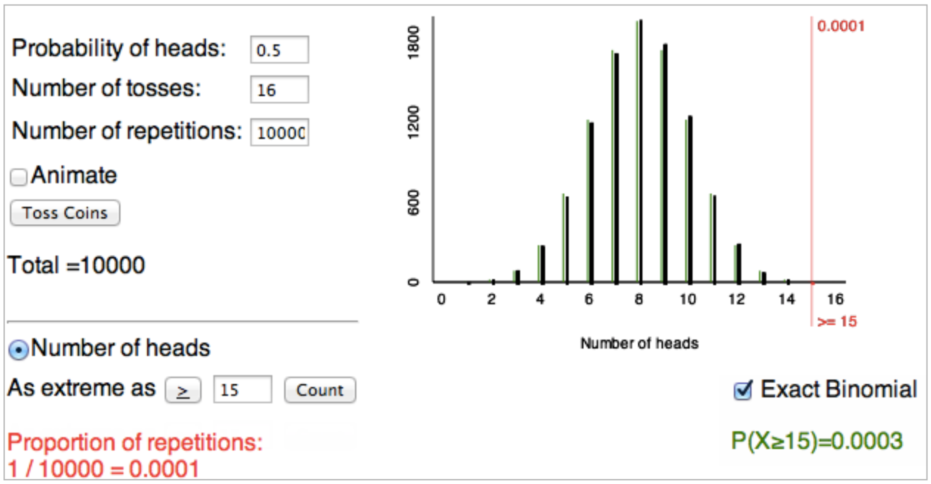
$$= np$$

Binomial and Sign Tests

24. Do you remember Buzz and Doris, the dolphins from activity 1? Me neither, so let's have a quick recap:

    Buzz was shown a light and needed to communicate to Doris which switch to push in order to get a fish. <u>Of the 16 trials, Buzz and Doris "communicated" to push the correct switch 15 times</u>.

    In activity #1, we used a simulation technique to calculate the likelihood of Buzz and Doris getting at least 15 correct if, in fact, they aren't able to communicate. From this simulation, we found the likelihood was around 0.0001 (which, we concluded, provides evidence that Buzz and Doris CAN communicate).



    The simulation applet also was able to calculate an <u>exact binomial of 0.0003</u>. We're going to replicate that here.

    Write out the null and alternate hypotheses for this dolphin study.

    H_0: ____

    H_1: ____

    If the null hypothesis were true, what's the probability that Buzz and Doris would push the correct switch on any single attempt?

    If the null hypothesis were true, what's the likelihood that Buzz and Doris would have pushed the correct switch at least 15 times out of 16?

25. You're interested in determining whether homing pigeons really know how to return home. You blindfold 12 pigeons and drive them 20 miles due west. One at a time, you release the pigeons and look at the direction they begin to fly. If a pigeon begins to fly more eastward than westward, you will conclude that the pigeon really does know the way home.

Write out the null and alternate hypotheses for this study. Try to write them in terms of probabilities.

H_0: ____

H_1: ____

Under the null hypothesis, how many of our 12 pigeons would we expect to fly eastward? __________

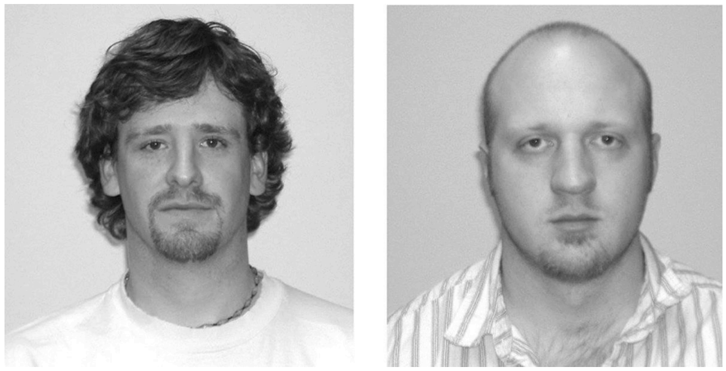
Assuming the null hypothesis is true, what's the likelihood of observing <u>10 or more pigeons flying eastward</u>?

As we've seen before, we call this likelihood a <u>p-value</u>. It represents the probability of observing results as extreme or even more extreme than our actual results if the null hypothesis is true. Based on this p-value, what would you conclude about the homing pigeons?

Suppose our experiment resulted in only 8 of the 12 pigeons flying eastward. Would you still make the same conclusion? Calculate a p-value.

26. A 2007 study presented evidence that "people use facial prototypes when they encounter different names." Participants were shown the following photo and were asked to identify which one was Tim and which one was Bob. The researchers wrote that their participants "overwhelmingly agreed" on which face belonged to Tim and which face belonged to Bob, but did not provide the exact results of their study.

Source: Lea, Thomas, Lamkin, & Bell, 2007. Psychonomic Bulletin and Review



Identify which one is Tim and which one is Bob.

How many students in this class chose the same face for Bob? Estimate the likelihood of this result (or something more extreme) if we assume students randomly assign names to faces.

27. Suppose we're interested in the effects of a drug on blood pressure levels. To study this, we get 10 middle-aged men to take the drug for 3 months. The change in blood pressure for each subject is displayed in this table:

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Change in Blood Pressure	-5	+2	+10	-8	-6	-10	-5	+2	-5	-8

We could also choose to ignore the magnitude of change and simply note:

Change in Blood Pressure	–	+	+	–	–	–	–	+	–	–

Under a null hypothesis, what should we expect for the probability of observing a + (or a –)? Under this null hypothesis, what's the likelihood that we would have observed data as or more extreme than what we observed? What conclusion can we make from this?

29. Do dogs resemble their owners? To study this, researchers took photos of 45 dogs and their owners. They then constructed triads of the pictures, each consisting of (a) the dog owner, (b) that owner's dog, and (c) a photo of another dog.

Not an actual example from this study: 

The researchers then showed these photo triads to a class of 28 psychology students. The students were asked to identify which dog belonged to the owner. <u>If more than half (more than 14) of the students chose the correct dog, then the dog was regarded as resembling the owner</u>.

Out of the 20 nonpurebred dogs in the study, 7 were found to resemble their owners.

Out of the 25 purebred dogs, 16 were found to resemble their owners.

Source: Levine, D.W. (2005). Do Dogs Resemble Their Owners? A Reanalysis of Roy & Christenfeld. Psychological Science, 16(1), 83-84. http://psy2.ucsd.edu/~nchristenfeld/publications_files/dogs.pdf

Let's assume dogs do NOT resemble their owners (and, therefore, the students were just randomly guessing which of the two dogs belonged to the owner). For any photo triad, what's the probability more than 14 students would choose the correct dog?

Given your answer to the previous question, what was the likelihood of observing 7 or more nonpurebred dogs that resemble their owners? What was the likelihood of observing 16 or more purebred dogs that resemble their owners?