Solutions to ANOVA exercises

NOTE: You can use an online ANOVA calculator at http://www.danielsoper.com/statcalc/calc43.aspx (google "ANOVA calculator")

1) Decide whether or not an ANOVA is appropriate.

a) Are we comparing two or more group (treatment) means?

b) Does our data meet the assumptions necessary to run an ANOVA?

   a. Are the groups independent?

   b. Are the observations within each group normally distributed? (Graph, p-p plot, or use Kolmogorov-Smirnov test)

   c. Homogeneity of variance assumption: Are the group/treatment variances equal?

      i. Use the F_{max} test to check for equal variances:

Compute: $F_{\max} = \dfrac{\sigma^2_{largest}}{\sigma^2_{smallest}}$ and compare to $F^{\,a}_{\max\, n-1}$ from the F_{max} table, where a = # of groups and n = observations per group.

If your calculated F_{max} < the F_{max} value from the table, you can procede.
If your calculated F_{max} > the F_{max} value from the table, we cannot conduct an ordinary ANOVA.

c) State the null and alternate hypotheses: $H_0 : \mu_1 = \mu_2 = ... = \mu_a$ vs. H_A: Not H_0

or

$H_0 : \alpha_j = 0$ vs. $H_A : \alpha_j \neq 0$

d) Calculate the sample means and standard deviations for each treatment: $\overline{X}_a = \dfrac{1}{n_a}\sum x_{ia}$ and $s_a = \sqrt{\dfrac{\sum\left(x_{ia} - \overline{X}_a\right)^2}{n_a - 1}}$

e) Calculate the overall mean: $M = \overline{\overline{X}} = \dfrac{\sum n_a \overline{X}_a}{N}$

f) Calculate sums of squares: $SS_A = \sum n_a\left(\overline{X}_a - M\right)^2$ $\quad SS_E = \sum (n_a - 1)s_a^2$ $\quad SS_T = SS_A + SS_E$

g) Calculate degrees of freedom: $df_A = a - 1$ $\quad df_E = N - a$ $\quad df_T = N - 1$

h) Fill-in the ANOVA summary table with your calculated SS and df values:

Source	Sums of Squares	Degrees of freedom	Mean Square	Mean Square Ratio
Between Groups (Treatment Effect)				
Within Groups (Error variance)				
Total				

i) Calculate the mean squares: $MS_A = \dfrac{SS_A}{df_A}$ $\quad MS_E = \dfrac{SS_E}{df_E}$

j) Calculate the mean square ratio: $MSR = \dfrac{MS_A}{MS_E}$ and compare it to F^{a-1}_{N-a} from the F-tables.

If MSR > F^{a-1}_{N-a} we reject the null hypothesis. If MSE < F^{a-1}_{N-a} we retain the null hypothesis.

k) Calculate the effect size: $\eta^2 = \dfrac{SS_A}{SS_T}$ (the proportion of total variance accounted for by the treatments)

l) Write your conclusion and perform appropriate follow-up tests.

Exercise #1:

		Sugar	
	N	Mean	SD
Shelf #1	20	4.80	2.138
Shelf #2	20	9.85	1.985
Shelf #3	20	6.10	1.865

$H_0 : \alpha_j = 0$ (the three shelves have equal population means; no treatment effect)

$H_A : \alpha_j \neq 0$ (the treatment means are not all equal; a treatment effect exists)

$F_{\max} = \frac{\sigma^2_{largest}}{\sigma^2_{smallest}} = \frac{2.138^2}{1.865^2} = 1.314$ compared to $F^{3}_{\max 19} = 2.95$ (OK to move on)

$$M = \overline{\overline{X}} = \frac{(20)(4.8)+(20)(9.85)+(20)(6.10)}{60} = 6.9167$$

$$MS_A = \frac{SS_A}{df_A} = \frac{(20)(4.8-6.9167)^2+(20)(9.85-6.9167)^2+(20)(6.10-6.9167)^2}{3-1} = \frac{275.03}{2} = 137.515$$

$$MS_E = \frac{SS_E}{df_E} = \frac{(20-1)(2.138)^2+(20-1)(1.985)^2+(20-1)(1.865)^2}{60-3} = \frac{227.8}{57} = 3.996$$

Source	SS	df	MS	MSR
Shelf Effect	275.03	2	137.515	$\frac{137.515}{3.996} = 34.413$
Error	227.8	57	3.996	$F^2_{57} = 3.15$
Total	502.83	59		

Since our calculated (observed) ratio is bigger than the critical F-value, we reject the null hypothesis.

$$\eta^2 = \frac{275.03}{502.83} = 0.547$$

54.7% of the variance in sugar is accounted for by shelf location.

Exercise #2:

		Fiber	
	N	Mean	SD
Shelf #1	20	1.68	1.166
Shelf #2	20	0.95	1.162
Shelf #3	20	2.17	1.277

$H_0 : \alpha_j = 0$ (the three shelves have equal population means; no treatment effect)

$H_A : \alpha_j \neq 0$ (the treatment means are not all equal; a treatment effect exists)

$F_{\max} = \frac{\sigma^2_{largest}}{\sigma^2_{smallest}} = \frac{1.277^2}{1.166^2} = 1.208$ compared to $F^{3}_{\max 19} = 2.95$ (OK to move on)

$$M = \overline{\overline{X}} = \frac{(20)(1.68)+(20)(0.95)+(20)(2.17)}{60} = 1.6$$

$$MS_A = \frac{SS_A}{df_A} = \frac{(20)(1.68-1.6)^2+(20)(0.95-1.6)^2+(20)(2.17-1.6)^2}{3-1} = \frac{15.076}{2} = 7.538$$

$$MS_E = \frac{SS_E}{df_E} = \frac{(20-1)(1.166)^2+(20-1)(1.162)^2+(20-1)(1.277)^2}{60-3} = \frac{82.47}{57} = 1.447$$

Source	SS	df	MS	MSR
Shelf Effect	15.076	2	7.538	$\frac{7.538}{1.447} = 5.21$
Error	82.47	57	1.447	$F^2_{57} = 3.15$
Total	97.546	59		

Since our calculated (observed) ratio is bigger than the critical F-value, we reject the null hypothesis.

$$\eta^2 = \frac{15.076}{97.546} = 0.15$$

15% of the variance in fiber is accounted for by shelf location.
Is this of practical significance?

Exercise #3:

		Thickness	
	N	Mean	SD
Group 1	30	3.015	0.107
Group 2	30	3.018	0.155
Group 3	30	2.996	0.132

$H_0 : \alpha_j = 0$ (the three production lines produce glass of the same thickness)

$H_A : \alpha_j \neq 0$ (the production line effects the thickness of glass)

$F_{\max} = \dfrac{\sigma^2_{largest}}{\sigma^2_{smallest}} = \dfrac{0.155^2}{0.107^2} = 2.098$ compared to $F_{\max 29}^{\ 3} = 2.07$ (We shouldn't continue)

$$M = \overline{\overline{X}} = \frac{(30)(3.015) + (30)(3.018) + (30)(2.996)}{90} = 3.01$$

$$MS_A = \frac{SS_A}{df_A} = \frac{(30)(3.015-3.01)^2 + (30)(3.018-3.01)^2 + (30)(2.996-3.01)^2}{3-1} = \frac{0.00855}{2} = 0.004275$$

$$MS_E = \frac{SS_E}{df_E} = \frac{(30-1)(.107)^2 + (30-1)(.155)^2 + (30-1)(.132)^2}{90-3} = \frac{1.534}{87} = 0.0176$$

Source	SS	df	MS	MSR
Line	.0086	2	.004275	$\dfrac{.004275}{.0176} = 0.243$
Error	1.534	87	0.0176	$F^2_{87} = 3.09$
Total	1.5426	89		

Since our calculated (observed) ratio is smaller than the critical F-value, we retain the null hypothesis.

$$\eta^2 = \frac{.0086}{1.534} = 0.0056$$

0.56% of the variance in glass thickness is due to production line differences. You can see why there is no significant group effect.

Exercise #4:

		Insects	
	N	Mean	SD
Yellow	6	47.167	6.795
White	6	15.667	3.327
Green	6	31.5	9.915
Blue	6	14.833	5.345

$H_0 : \alpha_j = 0$ (the colors make no difference)

$H_A : \alpha_j \neq 0$ (the treatment means are not all equal; a treatment effect exists)

$F_{\max} = \dfrac{9.915^2}{3.327^2} = 9.19$ compared to $F_{\max 5}^{\ 4} = 13.7$ (OK to move on)

$$M = \overline{\overline{X}} = \frac{6[47.167 + 15.667 + 31.5 + 14.833]}{24} = 27.29$$

$$MS_A = \frac{SS_A}{df_A} = \frac{6\left[(47.167-27.29)^2 + (15.667-27.29)^2 + (31.5-27.29)^2 + (14.833-27.29)^2\right]}{4-1} = \frac{4218.54}{3} = 1406.18$$

$$MS_E = \frac{SS_E}{df_E} = \frac{(20-1)[6.795^2 + 3.327^2 + 9.915^2 + 5.345^2]}{24-4} = \frac{920.586}{20} = 46.03$$

Source	SS	df	MS	MSR
Color	4218.54	3	1406.18	$\dfrac{1406.16}{46.03} = 30.55$
Error	920.586	20	46.03	$F^3_{20} = 3.03$
Total	5139.126	23		

Since our calculated (observed) ratio is bigger than the critical F-value, we reject the null hypothesis.

$$\eta^2 = \frac{4218.54}{5139.126} = 0.82$$

82% of the variance in insects captured is accounted for by color.

Exercise #5:

Source	SS	df	MS	MSR
Groups	**476.88**	3	158.96	**2.53**
Error	**2009.92**	32	62.81	
Total	**2486.8**	**35**		

Source	SS	df	MS	MSR
Groups	**126.9534**	7	**18.1362**	5.01
Error	**79.64**	**22**	3.62	
Total	**206.5934**	29		

$$\eta^2 = \frac{476.88}{2486.8} = 0.19 \qquad \eta^2 = \frac{126.9534}{79.64} = 0.6145$$

Exercise #6:

Caffeine	Taps per minute	Mean	Std. Dev.
0 mg	242 245 244 248 247 248 242 244 246 242	244.8	2.394
100 mg	248 246 245 247 248 250 247 246 243 244	246.4	2.066
200 mg	246 248 250 252 248 250 246 248 245 250	248.4	2.214

Test of Homogeneity of Variances

TAPS

Levene Statistic	df1	df2	Sig.
.292	2	27	.749

ANOVA

TAPS

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	61.400	2	30.700	6.181	.006
Within Groups	134.100	27	4.967		
Total	195.500	29			

Multiple Comparisons

Dependent Variable: TAPS

	(I) CAFFENE	(J) CAFFENE	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
Tukey HSD	0 mg	100 mg	-1.60	.997	.261	-4.07	.87
		200 mg	-3.50*	.997	.004	-5.97	-1.03
	100 mg	0 mg	1.60	.997	.261	-.87	4.07
		200 mg	-1.90	.997	.156	-4.37	.57
	200 mg	0 mg	3.50*	.997	.004	1.03	5.97
		100 mg	1.90	.997	.156	-.57	4.37
Bonferroni	0 mg	100 mg	-1.60	.997	.360	-4.14	.94
		200 mg	-3.50*	.997	.005	-6.04	-.96
	100 mg	0 mg	1.60	.997	.360	-.94	4.14
		200 mg	-1.90	.997	.202	-4.44	.64
	200 mg	0 mg	3.50*	.997	.005	.96	6.04
		100 mg	1.90	.997	.202	-.64	4.44

*. The mean difference is significant at the .05 level.

Top secret information:

Let's take one more look at the data for exercise #6:

Caffeine	Taps per minute	Mean	Std. Dev.
0 mg	242 245 244 248 247 248 242 244 246 242	244.8	2.394
100 mg	248 246 245 247 248 250 247 246 243 244	246.4	2.066
200 mg	246 248 250 252 248 250 246 248 245 250	248.4	2.214

If we were to pool all 30 observations into a single group and calculate the variance, we would calculate our grand mean to be:

$$M = \frac{\sum_{i=1}^{n} X_i}{N} = \frac{242 + 245 + ... + 250}{30} = 246.5$$

We could also calculate a variance of our single group of 30 observations (a grand variance?) to be:

$$s^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1} = \frac{(242 - 246.5)^2 + ... + (250 - 246.5)^2}{30 - 1} = 2.596416629^2$$

What does this variance represent? It represents the **average variation in all our data**.

Recall our conceptualization of SS_{total}. We said it represents the ***total*** **variation in our data**. This implies that we can convert between our grand variance and SS_{total}.:

$$SS_{total} = \sum_{i=1}^{n} (X_i - M)^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 = (N-1)\left[\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{N-1}\right] = (N-1)s^2$$

So if we calculate (or if we're given) the variance for our entire data set, we can use that to calculate SS_{total}.

In this example:

$$SS_{total} = (N-1)s^2 = (30-1)(2.596416629^2) = 195.5$$